Chapter 2
Binomial Theorem

2.1 Introduction:
An algebraic expression containing two terms is called a binomial expression. Bi means two and nom means term. Thus the general type of a binomial is \( a + b \), \( x - 2 \), \( 3x + 4 \) etc. The expression of a binomial raised to a small positive power can be solved by ordinary multiplication, but for large power the actual multiplication is laborious and for fractional power actual multiplication is not possible. By means of binomial theorem, this work reduced to a shorter form. This theorem was first established by Sir Isaac Newton.

2.2 Factorial of a Positive Integer:
If \( n \) is a positive integer, then the factorial of ‘\( n \)’ denoted by \( n! \) or \( n \) and is defined as the product of \( n \) +ve integers from \( n \) to 1 (or 1 to \( n \))
i.e., \( n! = n(n - 1)(n - 2) \ldots \ldots 3.2.1 \)
For example,
\[ 4! = 4.3.2.1 = 24 \]
and \[ 6! = 6.5.4.3.2.1 = 720 \]
one important relationship concerning factorials is that
\( (n + 1)! = (n + 1) n! \) \[ \text{Equation (1)} \]
for instance,
\[ 5! = 5.4.3.2.1 \]
\[ = 5(4.3.2.1) \]
\[ 5! = 5.4! \]
Obviously, \( 1! = 1 \) and this permits to define from equation (1)
\[ n!=\frac{(n+1)!}{n+1} \]
Substitute 0 for \( n \), we obtain
\[ 0!=\frac{(0+1)!}{0+1}=\frac{1!}{1}=\frac{1}{1}=1 \]

2.3 Combination:
Each of the groups or selections which can be made out of a given number of things by taking some or all of them at a time is called combination.
In combination the order in which things occur is not considered e.g.; combination of \( a, b, c \) taken two at a time are \( ab, bc, ca \).
The numbers \( \binom{n}{r} \) or \( n \binom{c}{r} \)

The numbers of the combination of \( n \) different objects taken ‘\( r \)’ at a time is denoted by \( \binom{n}{r} \) or \( n \binom{c}{r} \) and is defined as,

\[
\binom{n}{r} = \frac{n!}{r! (n-r)!}
\]

E.g., \( \binom{6}{4} = \frac{6!}{4!(6-4)!} = \frac{6 \times 5 \times 4!}{4! \times 2!} = \frac{6 \times 5}{2 \times 1} = 15 \)

Example 1: Expand \( \binom{7}{3} \)

Solution: \( \binom{7}{3} = \frac{7!}{3!(7-3)!} \)

\[
= \frac{7.6.5.4!}{3.2.1.4!} = \frac{7.6.5}{3.2.1} = 35
\]

This can also be expanded as

\[
\binom{7}{3} = \frac{7.6.5}{3.2.1} = 35
\]

If we want to expand \( \binom{7}{5} \), then

\[
\binom{7}{5} = \frac{7.6.5.4.3}{5.4.3.2.1} = 21
\]

Procedure: Expand the above number as the lower number and the lower number expand till 1.

Method 2

For expansion of \( \binom{n}{r} \) we can apply the method:

a. If \( r \) is less than \( (n - r) \) then take \( r \) factors in the numerator from \( n \) to downward and \( r \) factors in the denominator ending to 1.
b. If \( n - r \) is less than \( r \), then take \( (n - r) \) factors in the numerator from \( n \) to downward and take \( (n - r) \) factors in the denominator ending to 1. For example, to expand \( \binom{7}{5} \) again, here \( 7 - 5 = 2 \) is less than 5, so take two factors in numerator and two in the denominator as,

\[
\binom{7}{5} = \frac{7.6}{2.1} = 21
\]

**Some Important Results**

(i). \( \binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \times n!} = 1 \)

(ii) \( \binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \times 0!} = \frac{n!}{n! \times 1} = 1 \)

(iii) \( \binom{n}{r} = \binom{n}{n-r} \)

For example

\[
\binom{4}{0} = \binom{4}{4} = 1 \quad \text{as} \quad \frac{4!}{0!(4-0)!} = \frac{4!}{4! \times 0!} = \frac{4!}{4! \times 0!} = 1 = 1
\]

\[
\binom{4}{3} = \binom{4}{1} = 4 \quad \text{as} \quad \frac{4!}{3! \times 1!} = \frac{4!}{1! \times 3!} = \frac{4 \times 3!}{3! \times 1!} = \frac{4 \times 3!}{1! \times 3!} = 4 = 4
\]

**Note:** The numbers \( \binom{n}{r} \) or \( \binom{n}{c_r} \) are also called binomial co-efficients

**2.4 The Binomial Theorem:**

The rule or formula for expansion of \( (a + b)^n \), where \( n \) is any positive integral power, is called binomial theorem.

For any positive integral \( n \)

\[
(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2} + \binom{n}{3}a^{n-3}b^2 \ldots \ldots
\]

\[
+ \binom{n}{r}a^{n-r}b^r \ldots \ldots \ldots \ldots \ldots + \binom{n}{n}b^n \quad \text{---------------(1)}
\]
or briefly, \((a + b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r\)

**Remarks:** The coefficients of the successive terms are \(\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}\) and are called **Binomial coefficients**.

**Note:** Sum of binomial coefficients is \(2^n\)

**Another form of the Binomial theorem:**

\[(a + b)^n = a^n + \frac{n}{1!} a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3} b^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} a^{n-r} b^r + \ldots + b^n\]

**Note:** Since,

\[\binom{n}{r} = \frac{n!}{r!(n-r)!}\]

So,

\[
\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \times n!} = 1
\]

\[
\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n(n-1)!}{1!(n-1)!} = \frac{n}{1!}
\]

\[
\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)!}{2!(n-2)!} = \frac{n(n-1)}{2!}
\]

\[
\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)(n-3)!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{3!}
\]

\[
\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)!}{r!(n-r)!} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} = \frac{n!}{n!(n-r)!} = \frac{n!}{n \times 0!} = \frac{n!}{n \times 1} = 1
\]
The following points can be observed in the expansion of \((a + b)^n\)
1. There are \((n + 1)\) terms in the expansion.
2. The 1\textsuperscript{st} term is \(a^n\) and \((n + 1)\)th term or the last term is \(b^n\)
3. The exponent of ‘a’ decreases from \(n\) to zero.
4. The exponent of ‘b’ increases from zero to \(n\).
5. The sum of the exponents of \(a\) and \(b\) in any term is equal to index \(n\).
6. The co-efficients of the term equidistant from the beginning and end of the expansion are equal as \(\left(\begin{array}{c} n \\ r \end{array}\right) = \left(\begin{array}{c} n \\ n - r \end{array}\right)\)

2.5 General Term:

The term \(\left(\begin{array}{c} n \\ r \end{array}\right) a^{n-r} b^r\) in the expansion of binomial theorem is called the General term or \((r + 1)\)th term. It is denoted by \(T_{r+1}\). Hence

\[ T_{r+1} = \left(\begin{array}{c} n \\ r \end{array}\right) a^{n-r} b^r \]

Note: The General term is used to find out the specified term or the required co-efficient of the term in the binomial expansion

Example 2: Expand \((x + y)^4\) by binomial theorem:
Solution:

\[
(x + y)^4 = x^4 + \binom{4}{1} x^{4-1} y + \binom{4}{2} x^{4-2} y^2 + \binom{4}{3} x^{4-3} y^3 + y^4 \\
= x^4 + 4x^3y + \frac{4 \times 3}{2 \times 1} x^2y^2 + \frac{4 \times 3 \times 2}{3 \times 2 \times 1} xy^3 + y^4 \\
= x^4 + 6x^3y + 4x^2y^2 + 4xy^3 + y^4
\]

Example 3: Expand by binomial theorem \(\left(a - \frac{1}{a}\right)^6\)
Solution:

\[
\left(a - \frac{1}{a}\right)^6 = a^6 + \binom{6}{1} a^{6-1} \left(-\frac{1}{a}\right)^1 + \binom{6}{2} a^{6-2} \left(-\frac{1}{a}\right)^2 + \binom{6}{3} a^{6-3} \left(-\frac{1}{a}\right)^3 + \\
\binom{6}{4} a^{6-4} \left(-\frac{1}{a}\right)^4 + \binom{6}{5} a^{6-5} \left(-\frac{1}{a}\right)^5 + \binom{6}{6} a^{6-6} \left(-\frac{1}{a}\right)^6 \\
= a^6 + 6a^5 \left(-\frac{1}{a}\right) + \frac{6 \times 5}{2 \times 1} a^4 \left(-\frac{1}{a^2}\right) + \frac{6 \times 5 \times 4}{3 \times 2 \times 1} a^3 \left(-\frac{1}{a^3}\right) + \frac{6 \times 5 \times 4 \times 3}{4 \times 3 \times 2 \times 1} a^2 \left(-\frac{1}{a^4}\right) + \frac{6 \times 5 \times 4 \times 3 \times 2}{5 \times 4 \times 3 \times 2 \times 1} a \left(-\frac{1}{a^5}\right) + \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} \left(-\frac{1}{a^6}\right)
\]
Example 4: Expand \( \left( \frac{x^2}{2} - \frac{2}{x} \right)^4 \)

Solution:

\[
\left( \frac{x^2}{2} - \frac{2}{x} \right)^4 = \left( \frac{x^2}{2} \right)^4 + \binom{4}{1} \left( \frac{x^2}{2} \right)^{4-1} \left( -\frac{2}{x} \right)^1 + \binom{4}{2} \left( \frac{x^2}{2} \right)^{4-2} \left( -\frac{2}{x} \right)^2 \\
+ \binom{4}{3} \left( \frac{x^2}{2} \right)^{4-3} \left( -\frac{2}{x} \right)^3 + \binom{4}{4} \left( \frac{x^2}{2} \right)^{4-4} \left( -\frac{2}{x} \right)^4
\]

\[
= \frac{x^8}{16} + 4 \left( \frac{x^2}{2} \right)^3 \left( -\frac{2}{x} \right) + \frac{4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1} \left( \frac{x^2}{2} \right)^2 \left( \frac{4}{x^3} \right) + \\
\frac{4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1} \left( \frac{x^2}{2} \right) \left( -\frac{8}{x^3} \right) + \frac{16}{x^4}
\]

\[
= \frac{x^8}{16} + \frac{x^8}{8} \cdot \frac{2}{x} + 6 \cdot \frac{x^4}{4} \cdot \frac{4}{x^2} - 4 \cdot \frac{x^2}{2} \cdot \frac{8}{x^3} + 16 \cdot \frac{16}{x^4}
\]

\[
= \frac{x^8}{16} - x^5 + 6x^2 - \frac{16}{x} + \frac{16}{x^4}
\]

Example 5: Expand \((1.04)^5\) by the binomial formula and find its value to two decimal places.

Solution:

\[
(1.04)^5 = (1 + 0.04)^5 \\
(1 + 0.04)^5 = (1)^5 + \binom{5}{1} (1)^{5-1} (0.04) + \binom{5}{2} (1)^{5-2} (0.04)^2 + \binom{5}{3} (0.04)^3 \\
+ \binom{5}{4} (1)^{5-4} (0.04)^4 + (0.04)^5
\]

\[
= 1 + 0.2 + 0.016 + 0.00064 + 0.0000128 + 0.0000001024
\]

\[
= 1.22
\]

Example 6: Find the eighth term in the expansion of \( \left( 2x^2 - \frac{1}{x^2} \right)^{12} \)
Solution: \[\left(2x^2 - \frac{1}{x^2}\right)^{12}\]

The General term is, \[T_{r+1} = \binom{n}{r} a^{n-r} b^r\]

Here \[T_8 = ? \quad a = 2x^2, \quad b = \frac{-1}{x^2}, \quad n = 12, \quad r = 7\]

Therefore, \[T_{7+1} = \binom{12}{7}(2x^2)^{12-7} \left(\frac{-1}{x^2}\right)^7\]

\[T_8 = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \left(\frac{-1}{x^2}\right)^7 \left(2x^2\right)^5\]

\[T_8 = 793 \times 32x^{10} \frac{(-1)^7}{x^{14}}\]

\[T_8 = -\frac{25344}{x^4}\]

Eighth term = \[T_8 = -\frac{25344}{x^4}\]

2.6 Middle Term in the Expansion \((a + b)^n\)

In the expansion of \((a + b)^n\), there are \((n + 1)\) terms.

Case I:

If \(n\) is even then \((n + 1)\) will be odd, so \(\left(\frac{n}{2} + 1\right)\)th term will be the only one middle term in the expansion.

For example, if \(n = 8\) (even), number of terms will be 9 (odd), therefore, \(\left(\frac{8}{2} + 1\right) = 5^{th}\) will be middle term.

Case II:

If \(n\) is odd then \((n + 1)\) will be even, in this case there will not be a single middle term, but \(\left(\frac{n + 1}{2}\right)\)th and \(\left(\frac{n + 1}{2} + 1\right)\)th term will be the two middle terms in the expansion.

For example, for \(n = 9\) (odd), number of terms is 10 i.e. \(\left(\frac{9 + 1}{2}\right)\)

th and \(\left(\frac{9 + 1}{2} + 1\right)\)th i.e. 5th and 6th terms are taken as middle terms and these middle terms are found by using the formula for the general term.
Example 7: Find the middle term of \( \left( 1 - \frac{x^2}{2} \right)^{14} \).

**Solution:**
We have \( n = 14 \), then number of terms is 15.

\[ \therefore \left( \frac{14}{2} + 1 \right) \text{ i.e. } 8^{\text{th}} \text{ will be middle term.} \]

\[ a = 1, \quad b = -\frac{x^2}{2}, \quad n = 14, \quad r = 7, \quad T_8 = ? \]

\[ T_{r+1} = \binom{n}{r} a^{n-r} b^r \]

\[ T_{7+1} = \binom{14}{7} (1)^{14-7} \left( -\frac{x^2}{2} \right)^7 = \frac{14!}{7!7!} \left(-1\right)^7 \frac{x^{14}}{2^7} \]

\[ T_8 = \frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7! (-1)}{7!7!} \cdot \frac{1}{128} \cdot x^{14} \]

\[ T_8 = -\frac{429}{16} x^{14} \]

Example 8: Find the coefficient of \( x^{19} \) in \( (2x^3 - 3x)^9 \).

**Solution:**
Here, \( a = 2x^3, \quad b = -3x, \quad n = 9 \)

First we find \( r \).

\[ \text{Since } T_{r+1} = \binom{n}{r} a^{n-r} b^r \]

\[ = \binom{9}{r} \left( 2x^3 \right)^{9-r} \left( -3x \right)^r \]

\[ = \binom{9}{r} 2^{9-r} \left(-3\right)^r x^{27-3r} . x^r \]

\[ = \binom{9}{r} 2^{9-r} \left(-3\right)^r . x^{27-2r} \text{ ................. (1)} \]

But we require \( x^{19} \), so put
\[ 19 = 27 - 2r \]
\[ 2r = 8 \]
\[ r = 4 \]
Putting the value of \( r \) in equation (1)
\[
T_{4+1} = \binom{9}{r} 2^{9-r} (-3)^4 x^{19}
\]
\[
= \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} \cdot 2^5 \cdot 3^4 x^{19}
\]
\[
= 630 \times 32 \times 81 x^{19}
\]
\[
T_5 = 1632960 x^{19}
\]
Hence the coefficient of \( x^{19} \) is 1632960

Example 9: Find the term independent of \( x \) in the expansion of
\[
(2x^2 + \frac{1}{x})^9.
\]

**Solution:**
Let \( T_{r+1} \) be the term independent of \( x \).

We have \( a = 2x^2, b = \frac{1}{x}, n = 9 \)

\[
T_{r+1} = \binom{n}{r} a^{n-r} b^r = \binom{9}{r} (2x^2)^{9-r} \left( \frac{1}{x} \right)^r
\]

\[
= \binom{9}{r} 2^{9-r} \cdot x^{18-2r} \cdot x^r
\]

\[
= \binom{9}{r} 2^{9-r} \cdot x^{18-3r} \ldots \ldots \ldots \ldots (1)
\]

Since \( T_{r+1} \) is the term independent of \( x \) i.e. \( x^0 \).
\( \therefore \) power of \( x \) must be zero.

i.e. \( 18 - 3r = 0 \Rightarrow r = 6 \)

put in (1)

\[
T_{r+1} = \binom{9}{6} 2^{9-6} \cdot x^0 = \frac{19}{16!3^2} \cdot 1
\]

\[
= \frac{3 \cdot 9 \cdot 8^4 \cdot 7 \cdot 6!}{6! \cdot 3 \cdot 2 \cdot 1} \cdot 8 \cdot 1 = 672
\]
Exercise 2.1

1. Expand the following by the binomial formula.
   
   (i) \((x + \frac{1}{x})^4\)  
   (ii) \(\left(\frac{2x}{3} - \frac{3}{2x}\right)^5\)  
   (iii) \(\left(\frac{x}{2} - \frac{2}{y}\right)^4\)  
   
   (iv) \((2x - y)^5\)  
   (v) \(\left(2a - \frac{x}{a}\right)^7\)  
   (vi) \(\left(\frac{x}{y} - \frac{y}{x}\right)^4\)  
   
   (vii) \((-x + y^{-1})^4\)

2. Compute to two decimal places of decimal by use of binomial formula.
   
   (i) \((1.02)^4\)  
   (ii) \((0.98)^6\)  
   (iii) \((2.03)^5\)

3. Find the value of
   
   (i) \((x + y)^5 + (x - y)^5\)  
   (ii) \((x + \sqrt{2})^4 + (x - \sqrt{2})^4\)

4. Expanding the following in ascending powers of \(x\)
   
   (i) \((1 - x + x^2)^4\)  
   (ii) \((2 + x - x^2)^4\)

5. Find
   
   (i) the 5th term in the expansion of \(\left(2x^2 - \frac{3}{x}\right)^{10}\)
   
   (ii) the 6th term in the expansion of \(\left(x^2 + \frac{y}{2}\right)^{15}\)
   
   (iii) the 8th term in the expansion of \(\left(\sqrt{x} + \frac{2}{\sqrt{x}}\right)^{12}\)
   
   (iv) the 7th term in the expansion of \(\left(\frac{4x}{5} - \frac{5}{2x}\right)^9\)

6. Find the middle term of the following expansions
   
   (i) \(\left(3x^2 + \frac{1}{2x}\right)^{10}\)  
   (ii) \(\left(\frac{a}{2} - \frac{b}{3}\right)^{11}\)  
   (iii) \(\left(2x + \frac{1}{x}\right)^7\)

7. Find the specified term in the expansion of
   
   (i) \(\left(2x^2 - \frac{3}{x}\right)^{10}\) : term involving \(x^5\)
(ii) \( \left(2x^2 - \frac{1}{2x}\right)^{10} \) : term involving \( x^5 \)

(iii) \( \left(x^3 + \frac{1}{x}\right)^7 \) : term involving \( x^9 \)

(iv) \( \left(\frac{x}{2} - \frac{4}{x}\right)^8 \) : term involving \( x^2 \)

(v) \( \left(\frac{p^2}{2} + 6q^2\right)^{12} \) : term involving \( q^8 \)

8. Find the coefficient of

(i) \( x^5 \) in the expansion of \( \left(2x^2 - \frac{3}{x}\right)^{10} \)

(ii) \( x^{20} \) in the expansion of \( \left(2x^2 + \frac{1}{2x}\right)^{16} \)

(iii) \( x^5 \) in the expansion of \( \left(2x^2 - \frac{1}{3x}\right)^{10} \)

(iv) \( b^6 \) in the expansion of \( \left(\frac{a^2}{2} + 2b^2\right)^{10} \)

9. Find the constant term in the expansion of

(i) \( \left(x^2 - \frac{1}{x}\right)^9 \) (ii) \( \left(\sqrt{x} + \frac{1}{3x^2}\right)^{10} \)

10. Find the term independent of \( x \) in the expansion of the following

(i) \( \left(2x^2 - \frac{1}{x}\right)^{12} \) (ii) \( \left(2x^2 + \frac{1}{x}\right)^9 \)

Answers 2.1

1. (i) \( x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4} \)

(ii) \( \frac{32}{243}x^5 - \frac{40}{27}x^3\frac{20}{3}x - \frac{15}{x} + \frac{135}{8x^3} - \frac{243}{32x^5} \)

(iii) \( \frac{x^4}{16} - \frac{x^3}{y^2} + \frac{6x^2}{y} - \frac{6x}{y^3} + \frac{16}{y^4} \)
(iv) \[32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5\]

(v) \[128a^7 - 448a^5x + 672a^3x^2 - 560ax^3 + 280 \frac{x^4}{a} - 84 \frac{x^5}{a^3} + 14 \frac{x^6}{a^5} - \frac{x^7}{a^7}\]

(vi) \[- \frac{x^8}{y^8} - 8 \frac{x^6}{y^6} + 28 \frac{x^2}{y^2} - 56 \frac{x^4}{y^4} + 70 - 56 \frac{y^2}{x^2} + 28 \frac{y^4}{x^4} - 8 \frac{y^6}{x^6} + \frac{y^8}{x^8}\]

(vii) \[x^4 - 4x^3y^{-1} + 6x^2y^{-2} - 4xy^{-3} + y^{-4}\]

2. (i) 1.14  (ii) 0.88  (iii) 34.47

3. (i) \(2x^5 + 20x^3y^2 + 10xy^4\)  (ii) \(2x^4 + 24x^2 + 8\)

4. (i) \(-4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8\)
   (ii) \(16 + 32x - 8x^2 - 40x^3 + x^4 + 20x^5 - 2x^6 - 4x^7 + x^8\)

5. (i) \(1088640x^8\)  (ii) \(\frac{3003}{32}x^{20}y^5\)  (iii) \(\frac{101376}{x}\)  (iv) \(\frac{10500}{x^3}\)

6. (i) \(1913.625x^5\)  (ii) \(-\frac{77a^6b^5}{2592} + \frac{77a^5b^6}{3888}\)  (iii) \(\frac{280}{x} + 560x\)

7. (i) \(-1959552x^5\)  (ii) \(-252x^5\)  (iii) \(35x^9\)  (iv) \(-112x^2\)
   (v) \(\frac{880}{9}\)  \(p^{16}q^8\)

8. (i) \(-1959552\)  (ii) 46590  (iii) 33.185  (iv) \(\frac{15}{2}a^{14}\)

9. (i) 84  (ii) 5

10. (i) 7920  (ii) 672

### 2.7 Binomial Series

Since by the Binomial formula for positive integer \(n\), we have

\[(a + b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n - 1)}{2!}a^{n-2}b^2 + \frac{n(n - 1)(n - 2)}{3!}a^{n-3}b^3 + \ldots + b^n\]  \hspace{1cm} (2)

put \(a = 1\) and \(b = x\), then the above form becomes:

\[(1 + x)^n = 1 + \frac{n}{1!}x + \frac{n(n - 1)}{2!}x^2 + \ldots + x^n\]

if \(n\) is \(-ve\) integer or a fractional number (-ve or +ve), then
\[(1 + x)^n = 1 + \frac{n}{1!}x + \frac{n(n - 1)}{2!}x^2 + \ldots \ldots \ldots \infty \quad (3)\]

The series on the R.H.S of equation (3) is called binomial series.
This series is valid only when \(x\) is numerically less than unity
i.e., \(|x| < 1\) otherwise the expression will not be valid.

**Note:** The first term in the expression must be unity. For example, when
\(n\) is not a positive integer (negative or fraction) to expand \((a + x)^n\),
we shall have to write it as, \((a + x)^n = a^n \left(1 + \frac{x}{a}\right)^n\) and then apply
the binomial series, where \(\left|\frac{x}{a}\right|\) must be less than 1.

### 2.8 Application of the Binomial Series; Approximations:

The binomial series can be used to find expressions approximately
equal to the given expressions under given conditions.

**Example 1:** If \(x\) is very small, so that its square and higher powers
can be neglected then prove that
\[
\frac{1 + x}{1 - x}\]

Solution:
\[
\frac{1 + x}{1 - x} \text{ this can be written as } (1 + x)(1 - x)^{-1}
= (1 + x)(1 + x + x^2 + \ldots \ldots \text{ higher powers of } x)
= 1 + x + x + \text{neglecting higher powers of } x.
= 1 + 2x
\]

**Example 2:** Find to four places of decimal the value of \((1.02)^8\)

Solution:
\[
(1.02)^8 = (1 + 0.02)^8
= (1 + 0.02)^8
= 1 + \frac{8}{1}(0.02) + \frac{8.7}{2.1}(0.02)^2 + \frac{8.7.6}{3.2.1}(0.02)^3 + \ldots
= 1 + 0.16 + 0.0112 + 0.000448 + \ldots
= 1.1716
\]

**Example 3:** Write and simplify the first four terms in the expansion
of \((1 - 2x)^{-1}\).

Solution:
\[
(1 - 2x)^{-1}
= [1 + (-2x)]^{-1}
\]
Using \((1 + x)^n = 1 + nx + \frac{n(n - 1)}{2!} x^2 + \ldots \ldots \ldots \)
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\[= 1 + (-1)(-2x) + \frac{(-1)(-1 - 1)}{2!} (-2x)^2 + \ldots\]

\[= 1 + 2x + \frac{(-1)(-2)}{2.1} (4x^2) + \frac{(-1)(-2)(-3)}{3.2.1} (-8x^3) + \ldots\]

Example 4: Write the first three terms in the expansion of \((2 + x)^3\)

Solution:

\[(2 + x)^{-3} = (2)^{-3} \left(1 + \frac{x}{2}\right)^{-3}\]

\[= (2)^{-3} \left[1 + (-3) \left(\frac{x}{2}\right) + \frac{(-3)(-3-1)}{2!} \left(\frac{x}{2}\right)^2 + \ldots\right]\]

\[= \frac{1}{8} \left[1 - \frac{3}{2}x + 3x^2 + \ldots\right]\]

Root Extraction:

The second application of the binomial series is that of finding the root of any quantity.

Example 5: Find square root of 24 correct to 5 places of decimals.

Solution:

\[\sqrt{24} = (25 - 1)^{1/2}\]

\[= (25)^{1/2} \left(1 - \frac{1}{25}\right)^{1/2}\]

\[= 5 \left(1 - \frac{1}{5^2}\right)^{1/2}\]

\[= 5 \left[1 + \frac{1}{2} \left(-\frac{1}{5^2}\right) + \frac{1}{2!} \left(\frac{1}{2} \cdot \frac{-1}{2!}\right) \left(-\frac{1}{5^2}\right)^2 + \ldots\right]\]

\[= 5 \left[\frac{1}{2.5^2} - \frac{1}{2.3.5^4} - \frac{1}{2.4.5^6} + \ldots\right]\]

\[= 5 \left[1 - (0.02 + 0.0002 + 0.000004 + \ldots)\right]\]
Example 6: evaluate $\sqrt[3]{29}$ to the nearest hundredth.
Solution:

$$\sqrt[3]{29} = (27 + 2)^{1/3} = \left[27 \left(1 + \frac{2}{27}\right)\right]^{1/3} = 3 \left[1 + \frac{2}{27}\right]^{1/3} + \ldots$$

$$= 3 \left[1 + \frac{1}{3} \left(\frac{2}{27}\right) + \frac{1}{3} \left(\frac{1}{3} - 1\right) \left(\frac{2}{27}\right)^2 + \ldots \ldots\right]$$

$$= 3 \left[1 + \frac{2}{81} + \frac{1}{2} \left(\frac{2}{3}\right) \left(\frac{2}{27}\right)^2 + \ldots \ldots\right]$$

$$= 3 \left[1 + 0.0247 - 0.0006 \ldots \ldots\right]$$

$$= 3 [1.0212] = 3.07$$

Exercise 2.2

Q1: Expand up to four terms.
(i) $(1 - 3x)^{1/3}$  (ii) $(1 - 2x)^{-3/4}$  (iii) $(1 + x)^{-3}$
(iv) $\frac{1}{\sqrt{1 + x}}$  (v) $(4 + x)^{1/2}$  (vi) $(2 + x)^{-3}$

Q2: Using the binomial expansion, calculate to the nearest hundredth.
(i) $\sqrt[4]{65}$  (ii) $\sqrt[7]{17}$  (iii) $(1.01)^{-7}$
(iv) $\sqrt[8]{28}$  (v) $\sqrt[40]{40}$  (vi) $\sqrt[80]{80}$

Q3: Find the coefficient of $x^5$ in the expansion of
(i) $\frac{(1 + x)^2}{(1 - x)^2}$  (ii) $\frac{(1 + x)^2}{(1 - x)^3}$

Q4: If $x$ is nearly equal to unity, prove that
$$\frac{m x^n - n x^m}{x^n - x^m} = \frac{1}{1 - x}$$

Answers 2.2

Q1: (i) $1 - x - x - \frac{5}{3} x^3 + \ldots$  (ii) $1 + \frac{3}{2} x + \frac{21}{8} x^2 + \frac{77}{16} x^3 + \ldots$
(iii) $1 - 3x + 6x^2 - 10^3 \ldots$  (iv) $1 - \frac{1}{2} x + \frac{3}{8} x^2 - \frac{5}{16} x^3 + \ldots$
(v) \[2 + \frac{x}{2} - \frac{x^2}{64} + \frac{x^3}{512} + \cdots\] (vi) \[\frac{1}{8}\left[1 - \frac{3}{2}x + \frac{3}{2}x^2 - \frac{5}{4}x^3\right]\]

Q2: (i) 2.84 (ii) 4.12 (iii) 0.93 (iv) 5.29 (v) 6.32 (vi) 8.94

Q3: (i) 20 (ii) 61

**Summary**

**Binomial Theorem**

An expression consisting of two terms only is called a binomial expression. If n is a positive index, then

1. The general term in the binomial expansion is \( T_{r+1} = \binom{n}{r} a^{n-r} b^r \)
2. The number of terms in the expansion of \((a + b)^n\) is \(n + 1\).
3. The sum of the binomial coefficients in the expansion of \((a + b)^n\) is \(2^n\), i.e. \(\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n\)
4. The sum of the even terms in the expansion of \((a + b)^n\) is equal to the sum of odd terms.
5. When n is even, then the only middle term is the \(\left(\frac{n+2}{2}\right)\) th term.
6. When n is odd, then there are two middle terms viz \(\left(\frac{n+1}{2}\right)\) th and \(\left(\frac{n+3}{2}\right)\) th terms.

Note: If n is not a positive index.

i.e. \((a + b)^n = a^n \left(1 + \frac{n}{a}\right)^n\)

\[= a^n \left[1 + n \left(\frac{b}{a}\right) + \frac{n(n-1)}{2!} \left(\frac{b}{a}\right)^2 + \cdots \right]\]

1. Here n is a negative or a fraction, the quantities \(\binom{n}{1}, \binom{n}{2}\) here no meaning at all. Hence co-efficients cannot be represented as \(\binom{n}{1}, \binom{n}{2}\).
2. The number of terms in the expansion is infinite as n is a negative or fraction.
Short Questions

Write the short answers of the following

Expand by Bi-nomial theorem Q.No. 1 to 4

Q.1 \((2x - 3y)^4\)
Q.2 \(\left(\frac{x}{y} + \frac{y}{x}\right)^4\)
Q.3 \(\left(\frac{x}{2} - \frac{2}{y}\right)^4\)
Q.4 \(\left(x + \frac{1}{x}\right)^4\)

Q5 State Bi-nomial Theorem for positive integer n
Q.6 State Bi-nomial Theorem for n negative and rational.

Calculate the following by Binomial Theorem up to two decimal places.

Q.7 \((1.02)^{10}\)
Q.8 \((1.04)^{5}\)

Q.9 Find the 7\(^{th}\) term in the expansion of \(\left(x - \frac{1}{x}\right)^9\)
Q.10 Find the 6\(^{th}\) term in the expansion of \((x + 3y)^{10}\)
Q.11 Find 5\(^{th}\) term in the expansion of \((2 x - \frac{x^2}{4})^7\)

Expand to three term

Q.12 \((1 + 2x)^{-2}\)
Q.13 \(\frac{1}{(1 + x)^2}\)
Q.14 \(\frac{1}{\sqrt{1+x}}\)
Q.15 \((4 - 3x)^{1/2}\)

Q.16 Using the Binomial series calculate \(\sqrt[3]{65}\) to the nearest hundredth.

Which will be the middle term/terms in the expansion of
Q.17 \((2x + 3)^{12}\)

Q.18 \((x + \frac{3}{x})^{15}\)

Q.19 Which term is the middle term or terms in the Binomial expansion of \((a + b)^n\)

(i) When \(n\) is even  
(ii) When \(n\) is odd

**Answers**

Q1. \(16x^4 - 96x^3y + 216x^2y^2 - 216 xy^3 + 81 y^4\)

Q2. \(\frac{x^4}{y^4} + 4\frac{x^2}{y^2} + 6 + 4\frac{y^2}{x^2} + 4\frac{y^4}{x^4}\)

Q3. \(\frac{x^4}{16} - \frac{x^3}{y} + \frac{6x^2}{y^2} - \frac{16x}{y^3} + \frac{16}{y^4}\)

Q4. \((x)^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}\)

Q5. \(1.22\)

Q6. \(1.22\)

Q7. \(\frac{35}{32}x^{11}\)

Q8. \(61236x^5y^5\)

Q9. \(\frac{84}{x^3}\)

Q10. \(1 - 4x + 12x^2 + \ldots\)

Q11. \(1 - 2x + 3x^2 + \ldots\)

Q12. \(1 - \frac{x}{2} + \frac{3}{8}x^2 + \ldots\)

Q13. \(2 - \frac{3x}{4} - \frac{9x^2}{64} + \ldots\)

Q14. \(4.02\)

Q15. \(1 - \frac{15}{7}(3)^7x\) and \(\frac{15}{8}(3)^8x\)

Q16. \(T_7 = \binom{12}{6}(2x)^6(3)^6\)

Q17. \(T_8 = \binom{15}{7}(3)^7x\) and \(T_9 = \binom{15}{8}(3)^8x\)

Q18. \(\binom{n}{2} + 1\) and \(\binom{n + 1}{2} + 1\)
Objective Type Questions

Q.1 Each question has four possible answers. Choose the correct answer and encircle it.

_1. Third term of \((x + y)^4\) is:
(a) \(4x^2y^2\) (b) \(4x^3y\) (c) \(6x^2y^2\) (d) \(6x^3y\)

_2. The number of terms in the expansion \((a + b)^{13}\) are:
(a) 12 (b) 13 (c) 14 (d) 15

_3. The value of \(\binom{n}{r}\) is:
(a) \(\frac{n!}{r!(n-r)!}\) (b) \(\frac{n}{r(n-r)}\) (c) \(\frac{n!}{r!(n-r)!}\) (d) \(\frac{n!}{(n-r)!}\)

_4. The second last term in the expansion of \((a + b)^7\) is:
(a) \(7a^6b\) (b) \(7ab^6\) (c) \(7b^7\) (d) 15

_5. \(\binom{6}{4}\) will have the value:
(a) 10 (b) 15 (c) 20 (d) 25

_6. \(\binom{3}{0}\) will have the value:
(a) 0 (b) 1 (c) 2 (d) 3

_7. In the expansion of \((a + b)^n\) the general term is:
(a) \(\binom{n}{r}a^r b^r\) (b) \(\binom{n}{r}a^{n-r} b^r\)
(c) \(\binom{n}{r-1}a^{n-r+1} b^{r-1}\) (d) \(\binom{n}{r}a^{n-r-1} b^{r-1}\)

_8. In the expansion of \((a + b)^n\) the term \(\binom{n}{r}a^{n-r} b^r\) will be:
(a) nth term (b) rth term (c) \((r + 1)\)th term (d) None of these

_9. In the expansion of \((a + b)^n\) the rth term is:
(a) \(\binom{n}{r}a^r b^r\) (b) \(\binom{n}{r}a^{n-r} b^r\)
10. In the expansion of \((1 + x)^n\) the co-efficient of 3\(^{rd}\) term is:

(a) \(\binom{n}{0}\)  (b) \(\binom{n}{1}\)  (c) \(\binom{n}{2}\)  (d) \(\binom{n}{3}\)

11. In the expansion of \((a + b)^n\) the sum of the exponents of a and b in any term is:

(a) \(n\)  (b) \(n - 1\)  (c) \(n + 1\)  (d) None of these

12. The middle term in the expansion of \((a + b)^6\) is:

(a) \(15a^4b^2\)  (b) \(20a^3b^3\)  (c) \(15a^2b^4\)  (d) \(6ab^5\)

13. The value of \(\binom{n}{n}\) is equal to:

(a) Zero  (b) 1  (c) \(n\)  (d) \(-n\)

14. The expansion of \((1 + x)^{-1}\) is:

(a) \(1 - x - x^2 - x^3 + \ldots\)
(b) \(1 - x + x^2 - x^3 + \ldots\)
(c) \(1 - \frac{x}{1!} - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots\)
(d) \(1 - \frac{1}{1!}x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \ldots\)

15. The expansion of \((1 - x)^{-1}\) is:

(a) \(1 + x + x^2 + x^3 + \ldots\)
(b) \(1 - x + x^2 - x^3 + \ldots\)
(c) \(1 + \frac{1}{1!}x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots\)
(d) \(1 - \frac{1}{1!}x + \frac{1}{2!}x^3 - \frac{1}{3!}x^3 + \ldots\)

16. Binomial series for \((1 + x)^n\) is valid only when:

(a) \(x < 1\)  (b) \(x < -1\)
(c) \(|x| < 1\)  (d) None of these

17. The value of \(\binom{2n}{n}\) is:

(a) \(\frac{2n}{n! \ n!}\)  (b) \(\frac{(2n)!}{n! \ n!}\)
(c) \frac{(2n)!}{n!} \quad (d) \frac{(2n)!}{n(n-1)!}

18. The middle term of \( \left( x - \frac{y}{x} \right)^4 \) is:

(a) \frac{4x^2}{y^2} \quad (b) 6 \quad (c) 8 \quad (d) \frac{4x}{y}

Answers

1. c 2. c 3. a 4. b 5. b
6. b 7. b 8. c 9. c 10. c
11. a 12. b 13. b 14. b 15. a
16. b 17. d 18. c 19. b 20. b